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Scale Analysis of Two-Fluid Relative Velocity Equation: (Evaluation of Drift-Flux Approximation)

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Background

In ***FLOW-3D***® a two-fluid system is described by a mixture velocity that is a volume-weighted average of the two fluids,

$$\vec{U} = f_1\vec{u}_1 + f_2\vec{u}_2, \tag{1}$$

and a relative velocity $\mathbf{u}_r = \mathbf{u}_1 - \mathbf{u}_2$ between the fluids that satisfies the equation

$$\frac{\partial \vec{u}_r}{\partial t} + \vec{u}_1 \bullet \nabla \vec{u}_1 - \vec{u}_2 \bullet \nabla \vec{u}_2 = -\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\nabla p - K\left(\frac{1}{\rho_1 f_1} + \frac{1}{\rho_2 f_2}\right)\vec{u}_r . \tag{2}$$

Subscripts refer to the microscopic densities, velocities and volume fractions of the two fluids. The drag coefficient K is defined such that $K\mathbf{u}_r$ is the force per unit volume on fluid 2 by fluid 1.

A simple relation exists between the two microscopic fluid velocities and the mixture and relative velocities,

$$\begin{aligned} \vec{u}_1 &= \vec{U} + f_2\vec{u}_r \\ \vec{u}_2 &= \vec{U} - f_1\vec{u}_r \\ f_1 + f_2 &= 1 \\ \rho &= \rho_1 f_1 + \rho_2 f_2 \end{aligned} \tag{3}$$

An often useful approximation, known as the "drift-flux" approximation, is to neglect the inertia terms on the left side of the relative velocity equation, Eq.2. This approximation assumes a balance between the first term on the right-hand side, the "buoyancy" force caused by pressure gradients, and the second term on the right-hand side, a resistance term caused by viscous drag arising from the relative velocity of the two fluids. In general, the drag coefficient K may be a function of the magnitude of the relative velocity, for example a Reynolds number dependent viscous resistance.

The drift-flux approximation is simple and convenient because it does not require the use of additional arrays to store the three components of relative velocity at two time-levels. But the question then arises as to the conditions under which this approximation gives acceptable accuracy. One way that may provide some guidance is to perform a scaling analysis to assess the relative importance of the different terms appearing in Eq.2.

Scaling

Suppose that L represents a characteristic length for significant variations in the relative velocity $\mathbf{u}_r$, and that T is a corresponding characteristic time. The orders of magnitude of terms in Eq.2 are then estimated to be:

$$\frac{u_r}{T}+\frac{u_r^2}{L}\approx\left[\frac{(\rho_2-\rho_1)\rho}{\rho_1\rho_2}\right]G+\left[\frac{\rho}{\rho_1 f_1\rho_2 f_2}\right]Ku_r\,, \tag{4}$$

where G is a measure of the body acceleration, G~|grad(p)/ρ| and the quantity u_r stands for the magnitude of the relative velocity. This relation is not an equation; rather it simply indicates the relative order of magnitude of terms in Eq.2.

There are several preliminary, but useful, observations to make about Eq.4:

- The […] coefficients appearing in terms on the right-hand side retain the actual density and fluid fraction dependences, because these can vary considerably depending on the values of the densities and fluid fractions.

- The pressure gradient (i.e., first term on the right-hand side) is the only term independent of u_r, which means that it is the driving acceleration controlling the magnitude of u_r.

- The characteristic length scale L must be similar to the scale over which the mixture pressure gradients occur, since they are the driver for the relative velocity.

- If the time scale T is that associated with advection changes in u_r, then T~L/u_r and the two inertia-related terms on the left-hand side of Eq.4 have the same order of magnitude. These terms are quadratic in u_r, which means they may be thought of as corrections to the drift-flux approximation. If, however, the time scale T is shorter than that associated with advection, for example, because of a sudden or localized pressure gradient change, then the time derivative term may be larger. An evaluation of the conditions under which inertia terms can be safely ignored, that is, the conditions suitable for the drift-flux approximation, will be presented in the following section.

- If one of the fluids is dispersed, for example, as particles or small fluid blobs of average radius r_p, then a reasonable choice for the drag coefficient K is a combination of Stokes drag and form drag,

$$K=N_p\rho_c\frac{\pi}{2}\left(C_d+\frac{12}{R_e}\right)r_p^2u_r\,, \tag{5a}$$

  where N_p is the number of particles per unit volume, ρ_c is the density of the continuum fluid, C_d is a viscous drag coefficient of order unity and R_e is the particle Reynolds number, $R_e=r_pu_r/\nu_c$, where ν_c is the kinematic viscosity of the continuous fluid. Since the volume fraction of the particles, f_p, is the product of N_p times the volume of a single particle, this relation can also be written as,

$$K = \frac{3\rho_c f_p}{8}\left(C_d + \frac{12}{R_e}\right)\frac{u_r}{r_p} \,. \qquad (5b)$$

Scaling the Relative Velocity Magnitude

A comparison of terms in Eq.4 can be used to estimate the size of the relative velocity. For this purpose we shall suppose that fluid number 2 is the dispersed component while fluid 1 is the continuous component, so that $\rho_2=\rho_p$ and $f_1=f_c$. We shall consider two limiting cases.

No Viscous Shear Stress

With no viscous stress the only thing to balance a pressure gradient is advection so that we must have a relative velocity estimated by the relation

$$u_r^2 \approx \left|\frac{1}{\rho_c} - \frac{1}{\rho_p}\right| \rho L G \,. \qquad (6a)$$

In a hydrostatic pressure gradient G is equal to the acceleration of gravity. While pressure gradients associated with mixture flows are scaled such that $G \sim U^2/L$. In either case, a low density for either fluid component increases the magnitude of the relative velocity, which is not surprising since there is no viscous resistance to slow it down.

No Inertia Effects (Drift-Flux)

If all inertia effects are neglected there must be a balance between pressure gradient accelerations and viscous resistance. Using relation Eq.5b the relative velocity can be estimated by

$$u_r^2 \approx \frac{8}{3}\left(\frac{\rho_p - \rho_c}{\rho_c}\right)\left(\frac{f_c r_p}{\left(C_d + \frac{12}{R_e}\right)}\right) G \qquad (6b)$$

In this case the relative velocity is proportional to the size of the particles and inversely proportional to the continuum density. An asymmetry in density is easy to comprehend when one considers the situation of water drops in air versus air bubbles in water. It takes some time for water droplets in air to move when the surrounding air is set into motion, but almost no time for bubbles to adjust to a change in water motion.

Conditions for an Accurate Drift-Flux Approximation

Using the dispersed flow assumption implicit in the drag expression Eq.5b, some guidelines can be set for the adequacy of the drift-flux approximation. The necessary condition for accuracy of the drift-flux assumption is that the inertia terms in Eq.2 be much smaller than the viscous drag term, which can be expressed as an inequality

$$\frac{r_p}{L} \leq \left(\frac{\rho_c + f_p(\rho_p - \rho_c)}{(1 - f_p)\rho_p} \right) \frac{3}{8} \left(C_d + \frac{12}{R_e} \right). \qquad (7)$$

On the right-hand side the product of the last two term is of order unity or possibly much larger at small particle Reynolds numbers. In fact, at small particle Reynolds numbers the drift-flux approximation should be accurate independent of any other considerations. This is sensible since a small Reynolds number implies a close coupling of particles with the continuum fluid so there can be little or no relative velocity, hence, no inertia effects.

In typical situations where $r_p/L \ll 1$ we should expect small relative velocities and correspondingly small particle Reynolds numbers, so a drift-flux approximation should always be acceptable. It is useful, however, to look more closely at Eq.7 to see what, if any, exceptions there might be.

At higher particle Reynolds numbers the drift-flux condition will still be satisfied if the size of the first factor on the right-hand side of Eq.7 is relatively large. There are several cases to consider.

Cases where $\rho_p \sim \rho_c$

When densities of the two fluids are similar, say two liquids or dispersed solid particles in a liquid, then Eq.7 approximately reduces to,

$$\frac{r_p}{L} \leq \left(\frac{1}{1 - f_p} \right) \frac{3}{8} \left(C_d + \frac{12}{R_e} \right)$$

In this case it appears that a possible limit situation for an accurate drift-flux approximation would occur at high particle Reynolds numbers, $C_d \sim 0.5$ for spherical particles and f_p close to zero, a set of values that give an approximate lower value for the right-hand side of the inequality of about 0.19. This result suggests that when particle sizes (radii) approach 20% of the length scale over which significant pressure gradients exist and particle Reynolds numbers are large, then there could be potentially important inertial effects neglected when using a drift-flux approximation. On the other hand, it is difficult to imagine circumstances where this combination of limiting conditions would occur.

Cases where $\rho_p < \rho_c$:

For situations where the particle density is small with respect to the continuum, for example, air bubbles in water the right-hand side of Eq.7 can be quite large. This makes physical sense since air bubbles move relative to the water because of buoyancy balanced by viscous stresses, but they have almost no inertia. Thus, almost any particle size compared to the length scale L will be small compared to the right-hand side of Eq.7, ensuring the adequacy of the drift-flux approximation.

Cases where $\rho_p > \rho_c$:

When the particle density is much larger than that of the continuum, for example, water droplets in air, then Eq.7 essentially reduces to,

$$\frac{r_p}{L} \leq \left(\frac{f_p}{1 - f_p} \right) \frac{3}{8} \left(C_d + \frac{12}{R_e} \right).$$

It is not readily apparent in this case when the inequality is satisfied. For small particle volume fractions, for instance, it appears that a violation of the inequality may occur because the leading factor on the right side could be quite small. However, if a small particle volume fraction is the result of small particle sizes with respect to L, then the left side is by definition small and the Reynolds number on the right side would also be small and further enhance the inequality necessary for the drift-flux assumption. Consistent with this conclusion is the observation made from Eq.6b that small particle sizes are likely to have smaller relative velocities, which further improves the drift-flux approximation

The only limit in which the drift-flux approximation would seem to be in question is that of high particle Reynolds numbers and a small volume fraction of particles. Even in this limit the scale for significant pressure changes, L, could not be much larger than the particle size itself, which would be necessary for the drift-flux approximation to fail.

Conclusions

Based on the above rough estimates of relative velocities and their causes, it is concluded that the drift-flux approximation should be good for nearly all practical applications.

For this reason there is little to be gained by extending the relative velocity equation to include inertial terms.

If the present investigation were to be extended to obtain more concrete criteria, it is suggested that some thought be given to the form of the inertia terms, which involve the changes of the fluid velocity components along their direction of flow (e.g., $\mathbf{u}_1 * \text{grad}\mathbf{u}_1$). For example, a cloud of water droplets falling through air with a terminal speed would contribute no inertia contribution to Eq.2. If a horizontal air velocity were to deflect the droplets and viscous stresses are small, then the pressure change in the gas is that needed to maintain an incompressible flow as it encounters the flow blockage caused by the particles. This gradient develops over a region containing many particles so that the scale L would be much larger than a particle size.