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ADDITION OF WAVE TRANSMITTING BOUNDARY CONDITIONS TO THE *FLOW-3D*® PROGRAM

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INTRODUCTION

To numerically investigate the effects of wave interactions with structures, it is desirable to have an open boundary condition that minimizes the reflection of outgoing waves. This capability permits a reduction in the extent of the computing mesh needed for accurate computations.

Open boundaries in low speed flow calculations are difficult to define because they can, in principle, influence the entire computational region. For wave propagation problems, it is natural to seek a boundary condition that will allow outgoing waves to smoothly leave the computational mesh with minimum reflection. This problem is analogous to wave absorption in experimental wave tanks, where one wants to eliminate the reflection of waves from the downstream end of the tank. In these tanks a variety of techniques are used, but nearly all of them employ some sort of energy dissipation (e.g., porous beaches). Regardless of the method employed, the length of absorption must be at least as long as the longest waves to be trapped.

In a computational wave tank we are not limited by physical dimensions, so we can imagine a mathematical continuation of the flow in the tank beyond the end of the computed region. This is the spirit of the Sommerfeld radiation boundary condition, which is a simple mathematical continuation having the form of out-going waves:

$$\frac{\partial Q}{\partial t} + C\frac{\partial Q}{\partial x} = 0 \qquad (1)$$

where Q is any quantity, positive x is directed out of the boundary and C is the local phase speed of the assumed wave-like flow.

To use this idea as a boundary condition, the flow should consist of wave-like disturbances that are propagating toward the boundary. The waves don't have to be directed normal to the boundary, they don't have to be linear, and no assumption about wavelength is necessary. The mathematical expression in Eq.1 is simply a statement that any flow quantity Q at the boundary will translate across the boundary with speed C. If there is no time variation, this condition reduces to the "continuative" boundary condition of zero normal derivative

To use Eq.1 as a basis for an "outflow" boundary condition in a three-dimensional computation, we must apply it to every flow variable at every location on the outflow boundary. The phase speed C is unlikely to be a constant because different quantities will be moving in different directions (e.g., surface waves don't move with the same velocity as a fluid particle) and waves may strike a boundary at different times from different directions. To account for this, we make a further assumption that Eq. 1 is not only valid at the boundary, but also at one control volume removed from the boundary. The phase speed C is assumed to have the same value at both locations. This assumption allows us to use numerical approximations for Eq. 1 at two locations: one to determine the value of C and the other to compute a boundary value for the quantity Q.

In the next section we discuss in some detail the way in which Eq. 1 is developed into a new boundary condition for the ***FLOW-3D***® computational fluid dynamics program. The new condition can be used for incompressible fluid flows that are confined or have free surfaces. It can also be used for fully compressible flows. Justification for the approach that we have taken will be given not only by examples, but also by showing that the mathematical formulation we use has many desirable properties.

MATHEMATICAL/PHYSICAL CONSIDERATIONS

A good discussion of early attempts to develop boundary conditions to prevent wave reflection is given by I. Orlanski (1976). He points out that methods using some sort of energy absorption have the disadvantage of affecting the flow away from a boundary (i.e., a distance large enough to absorb the largest wavelength of interest). Also, absorption methods usually distort a flow that is tangential to the boundary even though it may have no reflecting components.

In an effort to devise a better boundary condition to prevent wave reflection Orlanski adopted the Sommerfeld radiation condition. While he was not the first person to do this, his contribution was to use the Sommerfeld condition twice: once to compute the phase speed and then a second time to compute the needed boundary values. Orlanski presented successful demonstrations of his method for internal waves in stratified flow and for the growth of instabilities in a shear flow.

As pointed out by Orlanski, the use of the Sommerfeld radiation condition has the distinct advantage that it only involves numerical information in the immediate vicinity of a boundary. In particular, only two grid points inside the domain are used to compute a boundary value at the first point outside the boundary. Furthermore, the Sommerfeld condition has no influence on flow components tangential to the boundary.

In this note we adopt the Orlanski idea, but implement it in a way that is both simpler and better from a mathematical standpoint. To understand the difference, we must first point out that Eq. 1 is a first-order wave equation that describes wave propagation in the positive x direction. Because it is first order, it requires only one initial condition to specify a starting condition. In contrast, a second-order wave equation requires two initial conditions and can have waves propagating in both the positive and negative directions. In this case, if only positively moving waves are wanted, initial conditions must be selected very carefully to insure zero amplitude for the negatively moving components.

Our scheme differs from Orlanski's in that we compute the phase speed from an approximation to Eq. 1 that is displaced one grid cell away from the boundary. Orlanski uses an approximation that is shifted in both space and time. The consequence of this double shift is that his final boundary condition expression involves three time levels. That is, he has converted the original first order Sommerfeld condition into a second order expression. This suggests that Orlanski's boundary condition has the potential to induce waves moving into the computational domain.

This is one reason we claim the method described here is superior; our boundary condition remains a first order expression having only positively moving components. Furthermore, the restriction to a first order expression simplifies the implementation of the condition in the ***FLOW-3D***® computer program because it requires no additional storage for more time levels of information at the boundary.

IMPLEMENTATION INTO FLOW-3D

The new outflow boundary condition used in ***FLOW-3D***® will be illustrated in terms of the right (maximum x) boundary of the computational region. Let i be the index of the spatial location on or outside the boundary where we need a value of some quantity q. This value is the boundary condition we are seeking. We assume q_i^n is a known value of q at this point at time level n. This value may have been used to compute n+1 time-level quantities in the interior of the grid. What we are seeking, then, is a condition that will give us the next value, q_i^{n+1}, which is needed to begin the next cycle of computation.

We begin with a second order (in space and time) difference approximation for the Sommerfeld condition, Eq.1,

$$\frac{1}{2}\left(q_i^{n+1}-q_i^n\right)+\frac{1}{2}\left(q_{i-1}^{n+1}-q_{i-1}^n\right)+\frac{c\delta t}{\delta x}\left[\frac{1}{2}\left(q_i^{n+1}-q_{i-1}^{n+1}\right)+\frac{1}{2}\left(q_i^n-q_{i-1}^n\right)\right]=0 \qquad (2)$$

The first two terms are time derivatives centered about n+1/2. One term is at location i and the other at i-1, so their average is centered about x location i-1/2. The last term contains an average of two x derivatives centered about i-1/2 at times n and n+1. This term is also centered about i-1/2 and n+1/2. In this expression we have assumed a uniform separation of spatial points δx and a time interval of δt.

Rearranging this expression,

$$q_i^{n+1}=q_{i-1}^n+\frac{(1-C)}{(1+C)}\left(q_i^n-q_{i-1}^{n+1}\right) \qquad (3)$$

where there is used the notation C=cδt/δx. Next we rewrite Eq.3 by shifting the i index one node to the left and solve for the coefficient involving C, which gives us the prescription for the phase speed, or in this case a function of the phase speed,

$$\frac{(1-C)}{(1+C)}=\frac{\left(q_{i-1}^{n+1}-q_{i-2}^n\right)}{\left(q_{i-1}^n-q_{i-2}^{n+1}\right)} \qquad (4)$$

Substituting the expression in Eq. 4 into Eq. 3 gives the needed boundary value for q_i^{n+1}. In order to use this expression, however, there are two limiting cases that must be considered.

Because we are only interested in out-going waves and want to suppress incoming flow, the value of the non-dimensional phase speed C must not be negative. Furthermore, computational stability requires that C be less than or equal to unity. From Eq. 4 we can see that restricting the allowable interval for C to (0,1) also restricts the allowable range of the quantity (1-C)/(1+C) to the same interval,

$$0 \leq \frac{1-C}{1+C} \leq 1 \qquad (5)$$

This completes the specification of the outflow boundary condition: it is Eqs. 3-4 with the restrictions given by Eq. 5.

Before validating this prescription, it is worthwhile to consider the limiting values of C in more detail. When C=1, a wave with phase speed c moves a distance of one grid interval in one time step. Putting C=1 into Eq. 3 we have,

$$q_i^{n+1} = q_{i-1}^{n},$$

which is the correct difference expression of this limit, a shift of the i-1 value to location i.

At the other extreme, C=0, Eq. 3 gives for the boundary value,

$$q_i^{n+1} = q_{i-1}^{n} + (q_i^{n} - q_{i-1}^{n+1})$$

This can be easily interpreted if we rewrite it in the form,

$$\frac{(q_i^{n+1} - q_i^{n})}{\delta t} + \frac{(q_{i-1}^{n+1} - q_{i-1}^{n})}{\delta t} = 0$$

Now we see when C=0, the boundary condition assigns a value to q such that the average time derivative of q at the boundary, at time level n+1/2, is zero. This is a nice feature because it implies that the boundary condition is trying to keep the average value of q constant when there are no out-going waves.

It is interesting to compare this last result with the original scheme of Orlanski in which the condition C=0 led to the result $q_i^{n+1}=q_i^{n-2}$. This assignment contains no information from within the computational region, a fact noted by Orlanski and which led him to suggest that a specified boundary value be used instead. Of course, specifying a value is contrary to the goal of an outflow condition, which responds to flow conditions within the grid.

Finally, we note that any Fourier wave with a component in the x direction, and arbitrary phase speed c, is a solution of Eq. 3 and exhibits no damping. The same cannot be said of Orlanski's method. Thus, the new outflow boundary scheme described here is seen to have several additional advantages over previous work.

VERIFICATION OF OPEN-BOUNDARY CONDITION

In this section we present several computational examples that show off the new boundary condition.

A. Transmission of cylindrical surge wave

Our first example is a simple surge wave generated by the collapse of a cylindrical column of fluid. Initially water is resting in a tank with depth 30 m and the tank has a radius of 500 m. At the center of the tank there is a cylindrical column of water of radius 100 m extending to a height of 40 m. The outflow boundary condition is set at the right side of the tank (i.e., of the computational region).

When the column of fluid falls it generates a surge wave traveling outward. Figure 1 shows the initial configuration, an intermediate time of 20 s just before the surge reaches the outflow boundary, and finally at 100 s. Qualitatively, the surge appears to pass smoothly through the outflow boundary. A better impression of how well the new boundary condition performs can be obtained from Fig. 2, which shows the average kinetic energy in the computational region as a function of time. We see that most of the energy leaves the computation with the initial passage of the wave. Secondary waves continue for a time, but they are also radiated away and the residual water in the tank steadily approaches a quiescence condition.

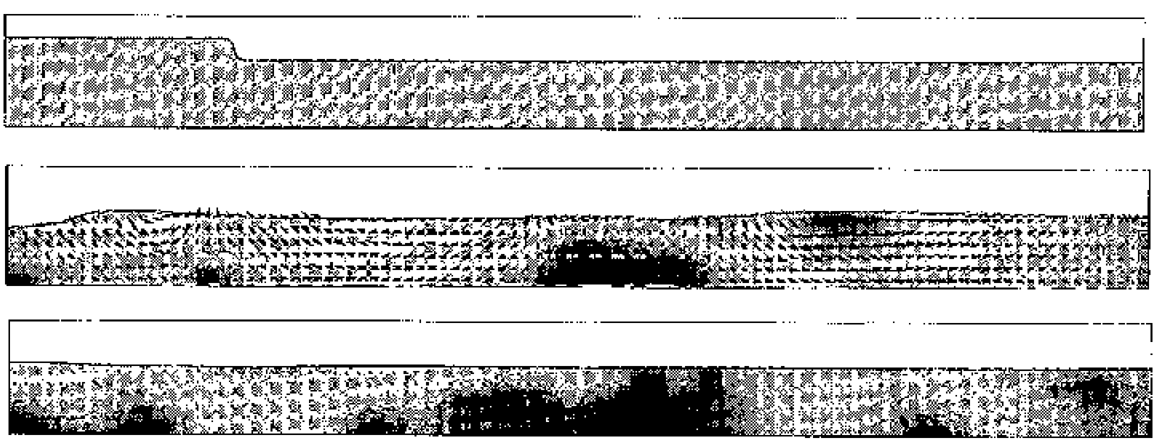

Figure 1. Surge produced by collapse of cylindrical column of water. Cylindrical axis is at left and outflow boundary is at right. Initial condition (top), 20 s just before surge leaves grid (middle), and 100 s (bottom).

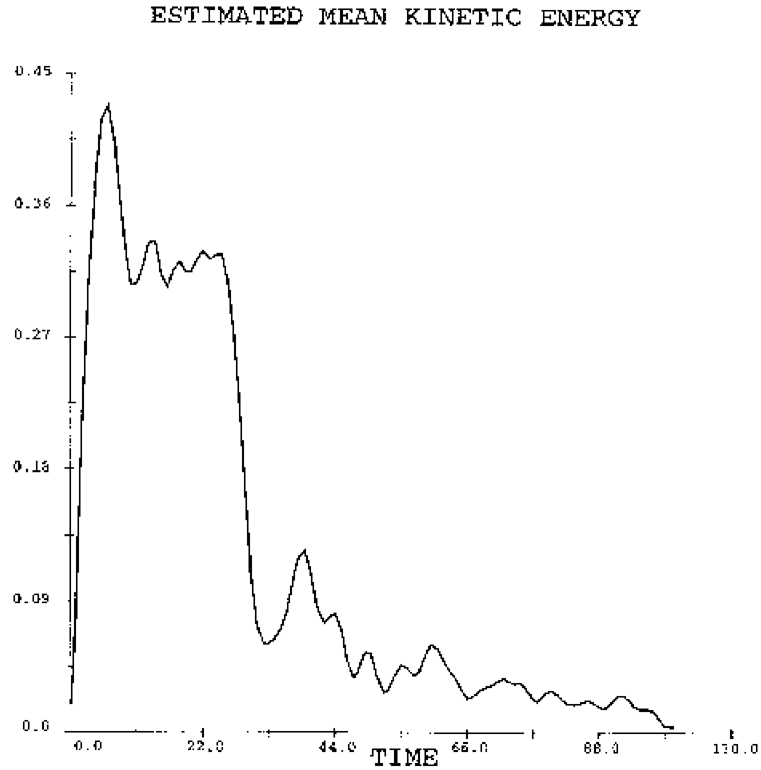

Figure 2. Time history of average kinetic energy in computational grid. Sharp drop between 20 and 30 s signals passage of surge wave through the outflow boundary. The energy goes steadily toward zero indicating a continuing radiation of waves.

B. Transmission of linear periodic surface waves

A generator of linear, periodic waves is placed at the left end of a wave tank of length 30 m that contains water to a depth of 8 m. The wave maker is set to generate waves having a wave number of 1.047 m^{-1}, a frequency of 3.2 s^{-1} and amplitude of 0.1 m. The outflow boundary condition is at the right side of the tank.

This configuration was used to run a large number of waves through the tank with no noticeable computational difficulties. To the eye there seemed to be no significant reflection of waves from the outflow boundary traveling back into the grid. Unfortunately, this does not provide a quantitative measure of the performance of the boundary conditions.

One means of getting a quantitative measure is to repeat the computation using a second grid that is 20% shorter than the first grid, 24 m instead of 30 m. If the outflow boundary condition is working properly, the flow computed in the short grid should agree with that computed in the corresponding region of the longer grid.

Figure 3 shows side-by-side comparisons taken from the two computations at the same times (only 24 m of the longer tank have been plotted to make comparisons easier). The agreement between the two cases after 60 s (approximately 30 waves) is extremely good, indicating that there is not significant reflection from the outflow boundary.

After 100 s (about 50 waves) there are small differences, which are most easily seen from a comparison of surface profiles as shown in Fig. 4. It is interesting to note that there was a difference in profiles at the wave generator, i.e., at the left boundary by 70 s. This observation suggests that a small perturbation has affected the generator and is then carried forward in time. Differences caused by such an event would be expected to grow linearly in time.

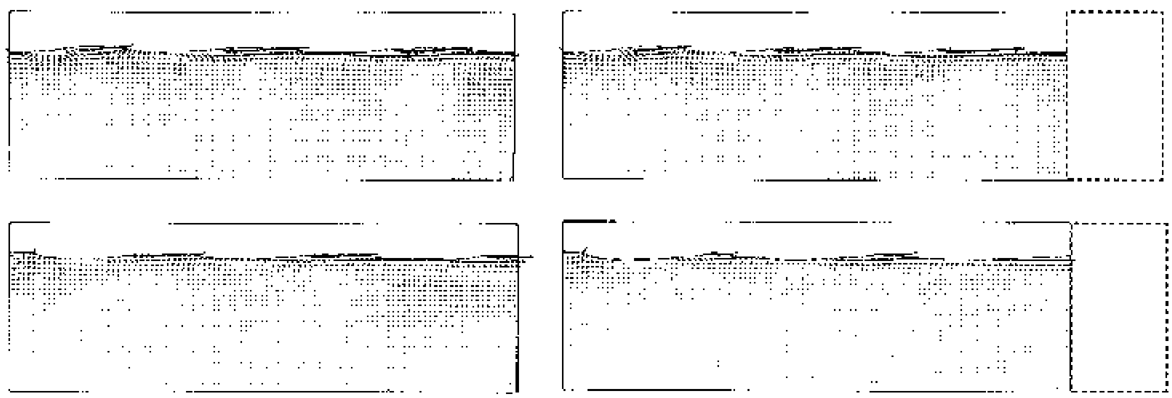

Figure 3. Comparison of short grid (left column) with longer grid (right column). Top plots at 60 s and bottom plots at 100 s. Small differences are seen by a time of 100 s.

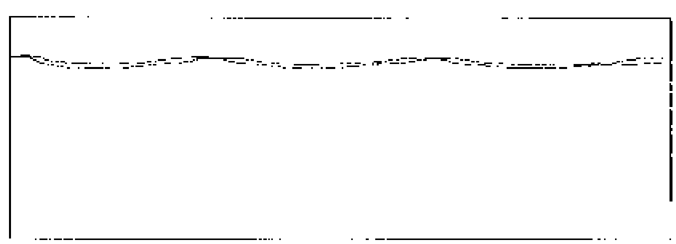

Figure 4. Comparison of short grid and long grid free surface profiles at 100 s, after about 50 wave periods.

C. Transmission of non-linear surface waves

The linear, periodic wave case described above was repeated using a non-linear wave generator provided by Dr. Sundar Prasad (1998). This case had shown poor performance when used to test an earlier attempt at an outflow boundary. With the present boundary model no computational difficulties were observed in 200 seconds of simulated time (about 30 waves). The outflow boundary condition performed smoothly and with no appearance of reflection, see Fig. 5, which shows the flow after nearly 30 waves have passed through the right boundary.

A gradual rise in fluid level was observed in this example, but it was traced to the wave generator and not to the new boundary condition.

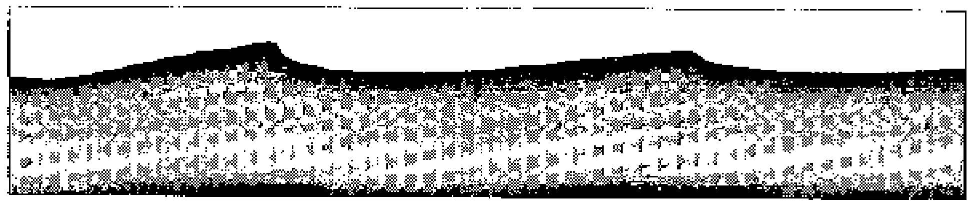

Figure 5. Computed flow of non-linear, periodic waves after nearly 30 waves have passed out the right outflow boundary.

D. Transmission of compressible waves

For completeness we include here a compressible flow example. The formulation of the outflow condition is independent of the level of compressibility of the fluid. The example we have selected is a shock-tube problem in which a diaphragm separates two states of a gas. To the left of the diaphragm the gas is at a higher pressure than the gas at the right. At time zero the diaphragm is removed causing a shock wave to move to the right and a rarefaction wave to move to the left. An outflow boundary is placed at the right end of the tube where the shock wave will leave the grid.

Figure 6 shows computed pressure profiles at three times during the computation: one before the shock reaches the right boundary, one immediately after the shock intersects the boundary and one showing a small rarefaction wave returning from the boundary. This reflected rarefaction is much smaller than the rarefaction wave that results when a continuitive boundary condition is used, Fig. 7.

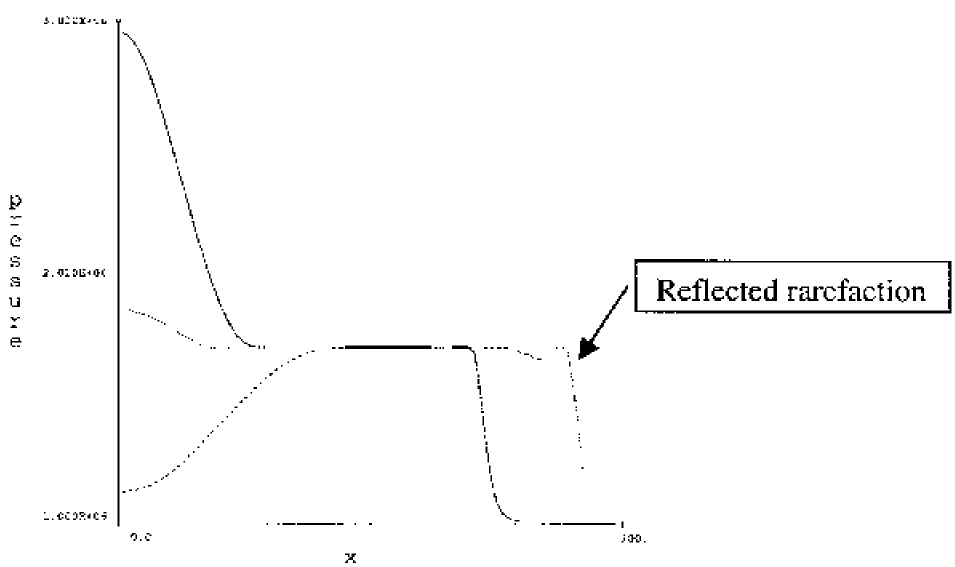

Figure 6. Computed pressure profiles for shock-tube problem. There is a small reflected rarefaction from outflow boundary at right.

The magnitude of the reflected wave can be greatly reduced by reducing the time-step size used in the computation (e.g., in Fig. 7 the smallest reflected wave that is plotted used a step size that was an order of magnitude smaller). The change in performance of the outflow boundary condition with time-step size is most likely associated with the steepness of the shock wave. Because the shock is so sharp, it contains components having very short wave lengths, which challenges the assumption of a constant phase speed near the boundary. A smaller time-step size helps in this regard since it gives the outflow conditions more opportunity (i.e., more steps) to adjust to the rapid changes occurring in the shock.

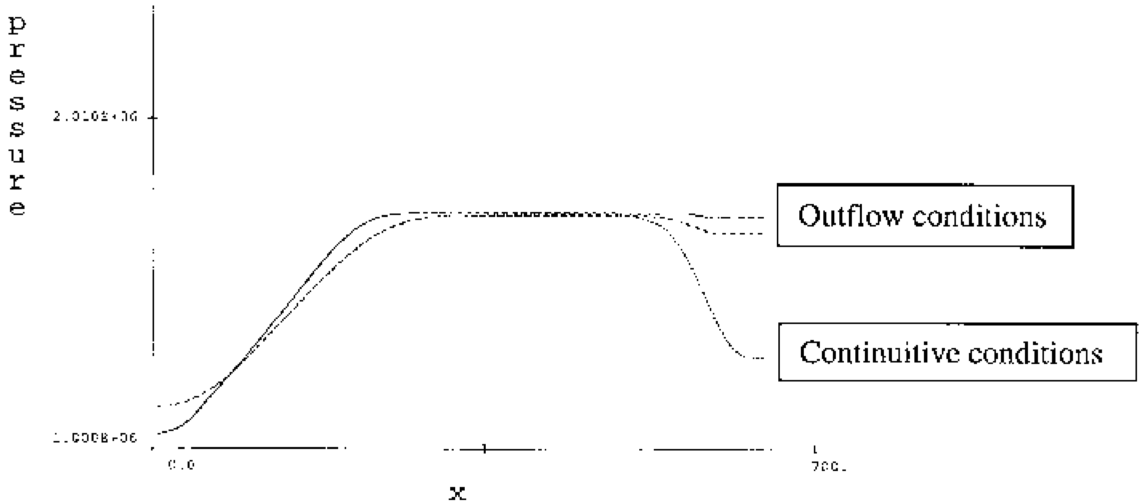

Figure 7. Comparison of reflected rarefaction waves. Outflow condition has minimum reflection and improves with smaller time-step sizes.

E. Multi-dimensional wave transmissions

The last example consists of waves generated by the collapse of a cylindrical column of water that intersect the sides of the computational region. A symmetry wall is located at the left side while the right side has the new outflow boundary condition. This example illustrates the transmission of waves that are not perpendicular to the boundary.

To show how the new boundary condition works under different conditions, we have used the shallow water approximation to model the flow in this example. The initial depth of fluid is 0.3m and the initial height of the cylindrical column is 0.4m. Width of the computed region is 10m.

Cylindrical waves moving out from the initial collapse after 1.5s are shown in the first plot of Fig. 8. At 3.5s the waves in the second plot of Fig. 8 have clearly reflected from the left boundary, but not from the right, outflow boundary. Even though the incident waves are circular and intersect the boundary in a continuously changing angle the new boundary condition is able to transmit them with little reflection.

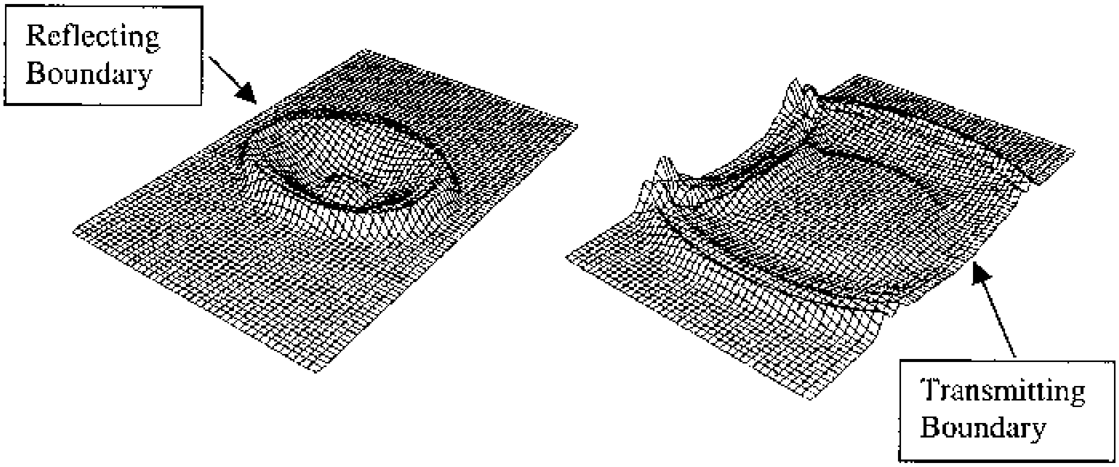

Figure 8. Collapse of cylindrical column of water in shallow pool. Transmitting boundary passes waves that intersect it at different angles. Surface disturbances have been amplified by a factor of 15 in the plots.

SUMMARY

A new outflow boundary condition has been developed that is based on the Sommerfeld radiation condition. This condition is designed to transmit outward propagating waves with little or no reflection. We have presented several mathematical reasons to support the approach we have taken. These include localization of the method to the immediate vicinity of the boundary, no damping of Fourier components and only outgoing wave contributions in the numerical scheme.

The new boundary condition has been illustrated with a variety of two and three-dimensional wave problems. The examples of wave transmission included linear and non-linear, free-surface waves as well as compressible shock waves. Furthermore, the outflow boundary condition was also demonstrated to work well with the shallow water (depth averaged) model.

This new boundary condition will be incorporated into the ***FLOW-3D***® program as an option (i.e., boundary option number 8) at all mesh boundaries.

REFERENCES

Orlanski, I., "A Simple Boundary Condition for Unbounded Hyperbolic Flows," J. Comp. Physics **21**, 251 (1976).

Prasad, S., Private communication (1998).